Solution, ordinary exam

Dec. 8, 2024 , 01001 Mathematics 1a

1. a

Q	P	$\neg Q$	$Q \Leftrightarrow P$	$\neg(Q \Leftrightarrow P)$	$\neg(Q \Leftrightarrow P) \wedge \neg Q$	$P \wedge \neg Q$
T	T	F	T	F	F	F
T	F	F	F	T	F	F
F	T	T	F	T	T	T
F	F	T	T	F	F	F

b

Same truth tables so $\neg(Q \Leftrightarrow P) \wedge \neg Q$ and $P \wedge \neg Q$ are logically equivalent.

2. Given: $p(z) = 2z^3 - 2z^2 - 8z - 12 \in \mathbb{C}[z]$ with root $z_0 = 3$.

a Polynomial division with $z-3$:

$$
\begin{array}{l}
z-3 \,|\, 2z^3 - 2z^2 - 8z - 12 \,|\, 2z^2 + 4z + 4 \\
\quad 2z^3 - 6z^2 \\
\quad\quad 4z^2 - 8z - 12 \\
\quad\quad 4z^2 - 12z \\
\quad\quad\quad 4z - 12 \\
\quad\quad\quad 4z - 12 \\
\quad\quad\quad\quad 0
\end{array}
$$

So, $p(z) = (z-3)(2z^2+4z+4)$, where $\deg(z-3) = 1$ and $\deg(2z^2+4z+4) = 2$.

b Roots in $p(z)$ are solutions to $p(z) = 0$, so using the rule of zero product, we solve the quadratic equation:

$$2z^2 + 4z + 4 = 0$$

$$\Updownarrow$$

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2} = -1 \pm \frac{\sqrt{16-32}}{4} = -1 \pm \frac{\sqrt{-16}}{4} = -1 \pm i$$

All roots of $p(z)$ are $\underline{\underline{-1+i}}$, $\underline{\underline{-1-i}}$, $\underline{\underline{3}}$

3.

$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in \mathbb{C}^{2\times 2}$ where $a \in \mathbb{C}$ is given

(a)

$$A^2 = AA = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$$

(b) Claim: $A^n = \begin{bmatrix} a^n & n a^{n-1} \\ 0 & a^n \end{bmatrix}$ for all $n \in \mathbb{Z}_{\geq 2}$

Showing via induction on n.

Base case, $n = 2$:

$A^2 = \begin{bmatrix} a^2 & 2a^{2-1} \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}$ fits, so base case is fulfilled.

Induction step, $n > 2$:

Induction hypothesis: $A^{n-1} = \begin{bmatrix} a^{n-1} & (n-1)a^{n-1-1} \\ 0 & a^{n-1} \end{bmatrix} = \begin{bmatrix} a^{n-1} & (n-1)a^{n-2} \\ 0 & a^{n-1} \end{bmatrix}$

$$A^n = A^{n-1}A = \begin{bmatrix} a^{n-1} & (n-1)a^{n-2} \\ 0 & a^{n-1} \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^{n-1}a & a^{n-1} + (n-1)a^{n-2}a \\ 0 & a^{n-1}a \end{bmatrix}$$

$$= \begin{bmatrix} a^{n-1+1} & a^{n-1} + (n-1)a^{n-2+1} \\ 0 & a^{n-1+1} \end{bmatrix} = \begin{bmatrix} a^n & a^{n-1} + (n-1)a^{n-1} \\ 0 & a^n \end{bmatrix}$$

$$= \begin{bmatrix} a^n & n a^{n-1} \\ 0 & a^n \end{bmatrix}$$, so fulfilled.

As per the induction principle, the claim is hence fulfilled for all $n \in \mathbb{Z}_{n \geq 2}$.

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4. $B = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 1 & 4 \\ 2 & -1 & 8 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

(a) Kernel: $[B \mid 0] = \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 2 & 1 & 4 & 0 \\ 2 & -1 & 8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & -7 & 14 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so $\ker B = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} t \mid t \in \mathbb{R} \right\}$ and a basis is $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(b) From the rref(B) above we see that the first and second columns are linearly independent, so a basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(c) For a linear map $L: \mathbb{R}^3 \to \mathbb{R}^3$, $L(v) = Bv$, the column space of B equals the image space of L, $\operatorname{colsp} B = \operatorname{image} L$. We have $\dim_{\mathbb{R}}(\operatorname{image} L) = \dim(\operatorname{colsp} B) = 2 \neq \dim_{\mathbb{R}}(\mathbb{R}^3) = 3$, so $\operatorname{image} L$ is not equal to the co-domain, and L is not surjective.

5. Given ordered basis for $V = \mathbb{R}^3$:

$\gamma = \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right)$ and β is the ordered standard basis

(a)

$${}_{\beta}\left[id_{\mathbb{R}^3} \right]_{\gamma} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\leftarrow$ γ-basis vectors in β-coordinates as columns

(b)

$${}_{\gamma}\left[id_{\mathbb{R}^3} \right]_{\beta} = \left({}_{\beta}\left[id_{\mathbb{R}^3} \right]_{\gamma} \right)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \tfrac{1}{2} \end{bmatrix}$$

The inverse:

$$\left[{}_{\beta}[id]_{\gamma} \mid \underline{\underline{I}}_3 \right] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \tfrac{1}{2} \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \tfrac{1}{2} \end{array} \right]$$

$$\underbrace{\begin{matrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \tfrac{1}{2} \end{matrix}}_{\left({}_{\beta}[id]_{\gamma} \right)^{-1}}$$

6.

$$\begin{cases} f_1'(t) = f_1(t) + 2f_2(t) \\ f_2'(t) = 2f_1(t) + f_2(t) \end{cases} \iff \begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

(a) No forcing functions, so the system is homogeneous.

(b)

Eigenvalues are found by solving the characteristic eq:

$$\det(\underline{A} - \lambda \underline{\underline{I}}_2) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0$$

$$\iff \lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = 1 \pm \frac{\sqrt{4+12}}{2} = 1 \pm \frac{4}{2}, \text{ so } \lambda_1 = 3, \lambda_2 = -1.$$

Eigenvector to $\lambda_1 = 3$:

$(\underline{A} - 3 \cdot \underline{\underline{I}}_2)\underline{v}_1 = \underline{0} \iff \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$, so an eigenvector is $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvector to $\lambda_2 = -1$:

$(\underline{A} - (-1)\underline{\underline{I}}_2)\underline{v} = \underline{0} \iff \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$, so an eigenvector is $\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

With two linearly independent eigenvectors $\underline{v}_1, \underline{v}_2$, (since they belong to different eigenspaces), then $\underline{\underline{A}}$ is diagonalizable, and the general, real-valued solution to the diff. eq. system is:

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

(c) Given initial-value condition: $f_1(0) = 3$ and $f_2(0) = 5$.

$$\begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = c_1 e^{3 \cdot 0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Updownarrow \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 1 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} c_1 = 4 \\ c_2 = 1 \end{cases}$$

So, the conditioned solution is:

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = 4e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$