EXAM, WRITTEN PART
01001/01003 Math 1a
May 2026, shsp

(9) Given $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(x) = x^2 - x$

a) $f(0) = 0^2 - 0 = \underline{\underline{0}}$, $f(1) = 1^2 - 1 = \underline{\underline{0}}$, $f(2) = 2^2 - 2 = 4 - 2 = \underline{\underline{2}}$

b) A function is injective if no two (different) inputs $x_1 \neq x_2$, where $x_1, x_2 \in \mathbb{R}_{\geq 0}$, have equal outputs. From a) we see that $x_1 = 0 \neq x_2 = 1 \Rightarrow f(x_1) = f(x_2) = 0$, so f is $\underline{\underline{\text{not}}}$ injective.

c) As f is not injective, it is also not bijective. Acc. to Lemma 2.2.2 f is hence $\underline{\underline{\text{not}}}$ invertible.

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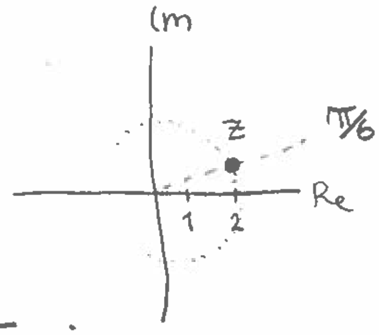
(10) Given $z = 2e^{i\pi/6}$



a) $\operatorname{mod} z = 2$, $\operatorname{Arg} z = \frac{\pi}{6}$

$\operatorname{Re} z = 2 \cdot \cos(\frac{\pi}{6}) = 2\frac{\sqrt{3}}{2} = \sqrt{3}$

$\operatorname{Im} z = 2 \cdot \sin(\frac{\pi}{6}) = 2\frac{1}{2} = 1$

Rectangular form: $z = \operatorname{Re} z + i \operatorname{Im} z = \underline{\underline{\sqrt{3} + i}}$

b) $z^6 = (2e^{i\frac{\pi}{6}})^6 = 2^6 e^{i\frac{\pi}{6}6} = 64 \overset{-1}{e^{i\pi}} = \underline{\underline{-64}}$ ← rectangular form with $\operatorname{Im} z^6 = 0$.

c) For z^n to be a positive real number, we can choose $n \in \mathbb{N}$ to be double the exponent in b), since that will double the argument from π to 2π.

$$z^{12} = (2e^{i\frac{\pi}{6}})^{12} = 2^{12} e^{i\frac{\pi}{6}12} = (2^6)^2 \overset{1}{e^{i2\pi}} = 64^2 \in \mathbb{R}_+$$

so, we choose $n = \underline{\underline{12}}$, and other choices would also work.

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(11) Given $\underline{\underline{C}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

a) The column space of $\underline{\underline{C}}$ over $\mathbb{R}$ is spanned by the columns of $\underline{\underline{C}}$, $\operatorname{colsp}_{\mathbb{R}} \underline{\underline{C}} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right)$, and a basis for it can be formed by the largest linearly independent subset of these col's.

$$\underline{\underline{C}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(\underline{\underline{C}})$$

With two pivots, we have rank $\rho(\underline{\underline{C}}) = \underline{\underline{2}}$ and thus $\dim \operatorname{colsp}_{\mathbb{R}} \underline{\underline{C}} = 2$. Hence a basis consists of two columns, and we see that the first two (with pivots) are linearly independent. An ordered basis for $\operatorname{colsp}_{\mathbb{R}} \underline{\underline{C}}$ can be $\underline{\underline{\left(\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}\right)}}$, but other choices are also possible.

b) Acc. to Corollary 8.2.5, $\det \underline{\underline{C}} \neq 0$ if and only if all columns of $\underline{\underline{C}}$ are linearly independent. As a) showed that the three col's of $\underline{\underline{C}}$ are linearly dependant, then $\det \underline{\underline{C}} = \underline{\underline{0}}$.

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12) Given $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \in V = \mathbb{R}^{2\times 2}$ over $\mathbb{R}$.

A map $L : V \to V$ is $L(\underline{\underline{M}}) = \underline{\underline{A}}\,\underline{\underline{M}} - \underline{\underline{M}}\,\underline{\underline{A}}$.

a) $L(\underline{\underline{0}}) = \underline{\underline{A}}\,\underline{\underline{0}} - \underline{\underline{0}}\,\underline{\underline{A}} = \underline{\underline{0}} - \underline{\underline{0}} = \underline{\underline{0}}$

$L(\underline{\underline{A}}) = \underline{\underline{A}}\,\underline{\underline{A}} - \underline{\underline{A}}\,\underline{\underline{A}} = \underline{\underline{0}}$

$$L(\underline{\underline{B}}) = \underline{\underline{A}}\,\underline{\underline{B}} - \underline{\underline{B}}\,\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 3 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

b) A linear map fulfills the linearity requirements of Def. 11.0.1

(i) $$L(\underline{\underline{U}} + \underline{\underline{V}}) = \underline{\underline{A}}(\underline{\underline{U}} + \underline{\underline{V}}) - (\underline{\underline{U}} + \underline{\underline{V}})\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{U}} + \underline{\underline{A}}\,\underline{\underline{V}} - \underline{\underline{U}}\,\underline{\underline{A}} - \underline{\underline{V}}\,\underline{\underline{A}}$$
$$= \underbrace{(\underline{\underline{A}}\,\underline{\underline{U}} - \underline{\underline{U}}\,\underline{\underline{A}})}_{L(\underline{\underline{U}})} + \underbrace{(\underline{\underline{A}}\,\underline{\underline{V}} - \underline{\underline{V}}\,\underline{\underline{A}})}_{L(\underline{\underline{V}})} = L(\underline{\underline{U}}) + L(\underline{\underline{V}})$$

(ii) $$c\,L(\underline{\underline{U}}) = c(\underline{\underline{A}}\,\underline{\underline{U}} - \underline{\underline{U}}\,\underline{\underline{A}}) = c\underline{\underline{A}}\,\underline{\underline{U}} - c\underline{\underline{U}}\,\underline{\underline{A}} = \underline{\underline{A}}(c\underline{\underline{U}}) - (c\underline{\underline{U}})\underline{\underline{A}} = L(c\underline{\underline{U}})$$

As L fulfills both for any $\underline{\underline{U}}, \underline{\underline{V}} \in V$ and $c \in \mathbb{R}$, then L is linear.

c) Given ordered basis for V: $\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

Acc. to Lemma 11.3.3, the mapping matrix ${}_{\mathcal{E}}[L]_{\mathcal{E}}$ has the mapped $\mathcal{E}$ basis vectors expressed in $\mathcal{E}$ basis as columns

$$L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \underline{\underline{A}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}, \quad \left[L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \left[L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}, \quad \left[L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad \left[L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So, we have $${}_{\mathcal{E}}[L]_{\mathcal{E}} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ -2 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

13 Given $\underline{M} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

a) $\underline{M}$ can be diagonalized to $\underline{D} = \underline{Q}^{-1}\underline{M}\,\underline{Q}$ by an invertible $\underline{Q}$ consisting of linearly independent eigenvectors (if enough exist) as columns, where the diagonal elements of $\underline{D}$ then will be the eigenvalues.

Solving the characteristic equation:

$$\det(\underline{M} - \lambda \underline{I}_2) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda) = 0 \iff \lambda = \begin{cases} 1 \\ 2 \end{cases}$$

The solutions are the eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$

Eigenvectors for $\lambda_1 = 1$:

$$(\underline{M} - 1\cdot\underline{I}_2)\underline{v}_1 = \underline{0} \iff \left[\begin{array}{cc|c} 1-1 & 1 & 0 \\ 0 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\iff \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t, \quad t \in \mathbb{R}$$

Eigenvectors for $\lambda_2 = 2$:

$$(\underline{M} - 2\cdot\underline{I}_2)\underline{v}_2 = \underline{0} \iff \left[\begin{array}{cc|c} 1-2 & 1 & 0 \\ 0 & 2-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\iff \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t, \quad t \in \mathbb{R}$$

So, $\underline{M}$ can be diagonalized to $\underline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ by $\underline{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

b) It is claimed that $P(n)$: $M^n = \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}$ for all $n \in \mathbb{N}$

Proving via induction on n:

Base case ($n=1$):

$$P(1): \begin{bmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \underline{M}^1 = \underline{M}$$

Fits, so base case holds

Induction hypothesis ($n>1$):

We assume $P(n-1)$: $M^{n-1} = \begin{bmatrix} 1 & 2^{n-1} - 1 \\ 0 & 2^{n-1} \end{bmatrix}$ for $n \in \mathbb{N}_{>1}$

Induction step ($n>1$)

$$P(n): \underline{M}^n = \underline{M}^{n-1}\underline{M} = \begin{bmatrix} 1 & 2^{n-1} - 1 \\ 0 & 2^{n-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 + 2^{n-1}\cdot 2 - 2 \\ 0 & 2^{n-1}\cdot 2 \end{bmatrix}$$

for $n \in \mathbb{N}_{>1}$

$$= \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}$$

Fits, so induction step holds

We conclude that the claim holds for all $n \in \mathbb{N}$.